

Sharp L^p -Boundedness of Oscillatory Integral Operators with Polynomial Phases

Zuoshunhua Shi ^{*} and Dunyan Yan [†]

Abstract

In this paper, we shall prove the L^p endpoint decay estimates of oscillatory integral operators with homogeneous polynomial phases S in $\mathbb{R} \times \mathbb{R}$. As a consequence, sharp L^p decay estimates are also obtained when polynomial phases have the form $S(x^{m_1}, y^{m_2})$ with m_1 and m_2 being positive integers.

Keywords: Sharp L^p boundedness, Oscillatory integral operators, Polynomial phases, Endpoint decay estimates.

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1 Introduction

In this paper, we mainly consider the following operator

$$T_\lambda(f)(x) = \int_{-\infty}^{\infty} e^{i\lambda S(x,y)} \varphi(x,y) f(y) dy, \quad (1.1)$$

where $\lambda \in \mathbb{R}$, $\varphi \in C_0^\infty(\mathbb{R}^2)$ and S is a real-valued homogeneous polynomial in $\mathbb{R} \times \mathbb{R}$. We can write the phase S as

$$S(x,y) = \sum_{k=0}^n a_k x^{n-k} y^k \quad (1.2)$$

with real coefficients a_k .

If S is a general real-valued smooth function with $S''_{xy} \neq 0$ on the support of the cut-off function, Hörmander ([12]) obtained the sharp L^2 operator norm estimate $\|T_\lambda\| \leq C|\lambda|^{-1/2}$. If the phase is degenerate on $\text{supp}(\varphi)$, then sharp decay estimates cannot be obtained directly and we need a suitable resolution of the singular variety $\{S''_{xy} = 0\}$. In this direction, the scalar oscillatory integrals with analytic phases were studied by Varchenko [32]. For homogeneous polynomials S , the study of these operators was initiated from [20]. In [21], Phong and Stein gave a necessary and sufficient condition under which the L^2 operator norm satisfies the sharp estimate $\|T_\lambda\| \leq C|\lambda|^{-\frac{1}{n}}$. For general real analytic phases, they established the relation between the sharp L^2 estimate and the Newton polyhedron of S ; see [22]. In [25], Rychkov extended L^2

^{*}School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, P. R. China.
E-mail address: shizuoshunhua11b@mails.ucas.ac.cn.

[†]School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100190, P. R. China.
E-mail address: ydunyan@ucas.ac.cn.

estimates in [22] to most smooth phases and full generalizations to smooth phases were proved by Greenblatt in [8]. For other related results, we refer the reader to [7], [10], [26] and [9].

Some averaging operators of Radon transforms are closely related to above oscillatory integral operators. Denote R by the Radon transform

$$Rf(x) = \int_{-\infty}^{\infty} f(x_1 + S(x_2, t), t) \varphi(x, t) dt$$

with $\varphi \in C_0^\infty(\mathbb{R}^2 \times \mathbb{R})$. The sharp $L^p - L^q$ estimates and L^p Sobolev regularity were obtained by Phong and Stein in [21] for homogeneous polynomials S except endpoint estimates. For analytic phases, endpoint $L^p - L^q$ estimates are previously known and sharp L^p Sobolev regularities were obtained except extreme points; see [14] and [31] as well as [1] and [2] by imposing certain left and right finite type conditions. It is notable that endpoint L^p Sobolev regularity may fail; see [5]. For more general Radon transforms which can be regarded as degenerate Fourier integral operators, we refer the reader to the survey paper [11] and references therein.

Our objective is to establish the sharp L^p estimates for T_λ . In the case of two sided fold singularities, Greenleaf and Seeger obtained in [10] the sharp L^p estimates for a smooth phase S . The sharp L^p estimates were obtained in [30] for homogeneous polynomial phases S under the assumption $a_1 a_{n-1} \neq 0$. For an analytic phase S , sharp L^p estimates of T_λ had been established in [31] except some extreme points of the reduced Newton polyhedron of S . Assume S is a real analytic function near the origin. Then the power expansion gives $S(x, y) = \sum_{k,l \geq 0} a_{k,l} x^k y^l$ near the origin. The Newton polyhedron is the convex hull of the sets $\{(x, y) : x \geq k, y \geq l\}$ with nonnegative integers k and l satisfying $a_{k,l} \neq 0$. Similarly, the reduced Newton polyhedron $\mathcal{N}(S)$ is obtained by taking the same convex hull with the additional condition $kl \neq 0$. Let T_λ be defined as in (1.1) with the cut-off function φ supported in a small neighborhood of the origin. If k and l are two positive integers such that $|\partial_x^k \partial_y^l S(0, 0)| \neq 0$, then the endpoint L^p estimates are given by

$$\|T_\lambda\|_{L^{(k+l)/k} \rightarrow L^{(k+l)/l}} \leq C |\lambda|^{-1/(k+l)}. \quad (1.3)$$

By interpolation, it is easily verified that these estimates imply the sharp L^2 decay rate obtained by Phong and Stein in [22]. We shall point out that the above estimates are sharp provided that (k, l) is a vertex of the reduced Newton polyhedron of S ; see [31]. If (k, l) is not a vertex but lying on the boundary of $\mathcal{N}(S)$, the above sharp estimates were obtained in [31]. But only weak type results were proved in [31] when (k, l) is a vertex. The question arises naturally whether the inequality (1.3) is true for extreme points (k, l) . In this paper, we shall give an affirmative answer when S is a real-valued homogeneous polynomial. Sharp estimates (1.3) are also true for a more general class of polynomials.

Now we turn to the issue for homogeneous phases S as in (1.2). To avoid triviality, we assume $a_k \neq 0$ for some $1 \leq k \leq n-1$. Define k_{min} and k_{max} as follows

$$k_{min} = \min \{1 \leq k \leq n-1 : a_k \neq 0\} \quad (1.4)$$

and

$$k_{max} = \max \{1 \leq k \leq n-1 : a_k \neq 0\}. \quad (1.5)$$

It is clear that $(k_{min}, n - k_{min})$ and $(k_{max}, n - k_{max})$ are two vertices of $\mathcal{N}(S)$.

Combining previously known results, we can state our main result as follows. We emphasize that only the endpoint estimates are new.

Theorem 1.1 Suppose that T_λ , S , k_{\min} and k_{\max} are given as above. Then the decay estimate

$$\|T_\lambda\|_{L^p \rightarrow L^p} \leq C|\lambda|^{-\frac{1}{n}} \quad (1.6)$$

holds for any amplitude $\varphi \in C_0^\infty$ if and only if $\frac{n}{n-k_{\min}} \leq p \leq \frac{n}{n-k_{\max}}$.

At the same time, we also consider the following two operators

$$T(f)(x) = \int_{-\infty}^{\infty} e^{iS(x,y)} f(y) dy \quad (1.7)$$

and

$$T_{m_1, m_2}(f)(x) = \int_{-\infty}^{\infty} e^{iS(x^{m_1}, y^{m_2})} f(y) dy \quad (1.8)$$

for $f \in C_0^\infty$ and positive integers m_1 and m_2 . By a routine scaling argument, L^p -boundedness of T is equivalent to the decay estimate (1.6). In the following sections, we shall prove Theorem 1.1 by showing that T is bounded on L^p for p in the range described as above. As a consequence, we also obtain the sharp L^p boundedness of T_{m_1, m_2} by invoking a simple interpolation lemma.

Theorem 1.2 Let m_1 and m_2 be two positive integers. Then T_{m_1, m_2} defined by (1.8) has a bounded extension from L^p to itself if and only if

$$\frac{k_{\min} m_2}{(n - k_{\min}) m_1} + 1 \leq p \leq \frac{k_{\max} m_2}{(n - k_{\max}) m_1} + 1.$$

Now we first present some previously known results. When $S(x, y) = c(x - y)^n$ for nonzero c , the L^p boundedness of T was established by [13] and [28] when $(x - y)^n$ is replaced by $|x - y|^n$; see also [15]. The arguments in the previous papers are also applicable for $S(x, y) = c(x - y)^n$. Another simple case is $S(x, y) = a_k x^{n-k} y^k$ with $a_k \neq 0$ and the corresponding result is contained in [17]. If S is not of the form $c(x - \alpha y)^n$, it was proved that (1.6) holds when $a_1 a_{n-1} \neq 0$ in [30]. For oscillatory integral operators with two sided fold singularities, we refer the reader to [10] for sharp L^p estimates. In [31], Yang obtained the sharp L^p decay estimates except the extreme points of $\mathcal{N}(S)$ when S is a real analytic phases.

Our proof of the main result relies on a complex interpolation between $H_E^1 - L^1$ and $L^2 - L^2$. This method appeared earlier in [19] and [10]. Here H_E^1 is a variant of Hardy spaces which will be defined later. To obtain sharp $H_E^1 - L^1$ and $L^2 - L^2$ estimates, we shall exploit a family of damped oscillatory integral operators. By insertion of the damping factor $|S''_{xy}|^{1/2}$ in (1.7), a useful sharp estimate has been obtained in [21]; see also [23] for the treatment of analytic phases. For convenience, we present the result in [21].

Theorem 1.3 ([21]) Assume that $S(x, y)$ is a real-valued homogeneous polynomial as in (1.2) with $a_k \neq 0$ for some $1 \leq k \leq n - 1$. Let U be given by

$$U(f)(x) = \int_{-\infty}^{\infty} e^{iS(x,y)} |S''_{xy}(x, y)|^{\frac{1}{2}} f(y) dy$$

for $f \in C_0^\infty$. Then U extends as a bounded operator from L^2 to itself.

It was remarked in [21] that the same result still holds if the damping factor $|S''_{xy}(x, y)|^{1/2}$ is replaced by $|S''_{xy}(x, y)|^\alpha$ with $\operatorname{Re}(\alpha) = 1/2$ for which the operator norm is bounded by a constant multiple of $(1 + |\operatorname{Im}(\alpha)|)^2$. This provides the sharp endpoint L^2 estimates. On the other hand, we shall see that the sharp $H_E^1 - L^1$ estimate (without decay on the parameter λ) is closely related to a class of oscillatory singular integral operators considered by Ricci and Stein in [24],

$$T_P(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,y)} K(x, y) f(y) dy, \quad (1.9)$$

where f is initially assumed to be smooth with compact support, $P(x, y)$ a real polynomial in $\mathbb{R}^n \times \mathbb{R}^n$ and $K(x, y)$ a Calderón-Zygmund kernel. It is known that T_P extends as a bounded operator from L^p to itself for $1 < p < \infty$ and the operator norm depends only on the degree of P but not on its coefficients; see [24] and [15]. In the case $p = 1$, T_P is bounded from H_E^1 to L^1 ; see [16]. Now we define the space H_E^1 , associated to a polynomial $P(x, y)$, using H_E^1 atoms ([16]).

Definition 1.1 Suppose that $P(\cdot, \cdot)$ is a real-valued polynomial in $\mathbb{R}^n \times \mathbb{R}^n$. Let $Q \subset \mathbb{R}^n$ be a cube with sides parallel to the axes and the center x_Q . An H_E^1 atom associated to the polynomial P is a measurable function a satisfying

$$(i) \operatorname{supp}(a) \subset Q; \quad (ii) |a(x)| \leq |Q|^{-1}, \text{ a.e. } x \in Q; \quad (iii) \int e^{iP(x_Q, y)} a(y) dy = 0.$$

Then $H_E^1(\mathbb{R}^n; P)$ consists of all those $f \in L^1(\mathbb{R}^n)$ which can be decomposed as $f = \sum_j \lambda_j a_j$, where each a_j is an atom associated to P and $\lambda_j \in \mathbb{C}$ satisfying $\sum_j |\lambda_j| < \infty$. The norm of f in H_E^1 is given by $\|f\|_{H_E^1} = \inf \left\{ \sum |\lambda_j| : f = \sum \lambda_j a_j, a_j \in H_E^1 \right\}$.

In the theory of singular integral operators, it is well known that the spaces H^1 and BMO are appropriate substitutes of L^1 and L^∞ , respectively. One aspect of this is that they play an important role in the interpolation of operators. A useful device is the sharp function invented by C. Fefferman and E. M. Stein in [6]. The dual space BMO_E of H_E^1 and the associated sharp function are defined as follows (see also [16] and [19]).

Definition 1.2 Let f_E^\sharp be the sharp function given by

$$f_E^\sharp(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q^E(y)| dy \quad (1.10)$$

with

$$f_Q^E(x) = e^{iP(x_Q, x)} \frac{1}{|Q|} \int_Q e^{-iP(x_Q, y)} f(y) dy.$$

Then BMO_E , associated to the polynomial P , consists of all locally integrable functions f such that $\|f\|_{BMO_E} = \|f_E^\sharp\|_{L^\infty} < \infty$. This finite number is defined to be the norm of f in BMO_E .

The present paper is organized as follows. §2 contains some basic lemmas and section 3 is devoted to mapping properties of some fractional integral operators. In §4, we shall treat oscillatory integral operators with polynomial phases and prove the estimate $H_E^1 - L^1$. The sharp L^2 decay estimates are obtained in §5 for damped oscillatory integral operators. The main result of Theorem 1.1 is proved in §6. In the final part §7, we shall apply Theorem 1.1 to prove Theorem 1.2 and give an easy proof of Pitt's inequality. The symbol C stands for a constant which may vary from line to line.

2 Some Basic Lemmas

The following lemma is useful in our proof of Theorem 1.2.

Lemma 2.1 *Let S be a sublinear operator which is initially defined for simple functions in \mathbb{R} . If there exists a constant $C > 0$ such that (i) $\|Sf\|_\infty \leq C\|f\|_1$ and (ii) $\|Sf\|_{p_0} \leq C\|f\|_{p_0}$ for some $1 < p_0 < \infty$ and all simple functions f , then the following inequality*

$$\int_{\mathbb{R}} |Sf(x)|^p |x|^{(p-p_0)/(p_0-1)} dx \leq C \int_{\mathbb{R}} |f(x)|^p dx$$

holds for $1 < p \leq p_0$, where the constant C is independent of f .

For the special case $p_0 = 2$, this lemma is contained in [17].

Proof. Let $Tf(x) = |x|^{1/(p_0-1)} Sf(x)$ and $d\mu = dx/|x|^{p_0/(p_0-1)}$, where dx denotes the Lebesgue measure. By the assumption (ii), we obtain that T is bounded from $L^{p_0}(\mathbb{R})$ to $L^{p_0}(\mathbb{R}, d\mu)$. Now we shall show that T maps $L^1(\mathbb{R})$ to $L^{1,\infty}(\mathbb{R}, d\mu)$. Indeed, we have

$$|Tf(x)| \leq |x|^{1/(p_0-1)} \|Sf\|_\infty \leq C|x|^{1/(p_0-1)} \|f\|_1.$$

For $\lambda > 0$, a simple calculation yields

$$\mu(\{x : |Tf(x)| > \lambda\}) \leq \frac{C}{\lambda} \|f\|_1. \quad (2.11)$$

The desired conclusion follows immediately from the Marcinkiewicz interpolation theorem. \square

The following lemma makes interpolation between $H_E^1 \rightarrow L^1$ and $L^2 \rightarrow L^2$ possible by the sharp function f_E^\sharp ; see Fefferman and Stein [6] and Phong and Stein [19].

Lemma 2.2 *If $F \in L^2$ and $F_E^\sharp \in L^p$ for some $2 \leq p < \infty$, then $F \in L^p$ and*

$$\|F\|_p \leq C_p \left\| F_E^\sharp \right\|_p.$$

Proof. Let f^\sharp be the well known sharp function defined by

$$f^\sharp(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where Q is a cube with sides parallel to the axes and f_Q the average of f over Q . Observe that $f^\sharp(x) \leq 2f_E^\sharp(x)$. By the assumption and the Fefferman-Stein theorem (see [6] or [29]) about sharp functions, we obtain $\|f\|_p \leq C\|f^\sharp\|_p \leq C\|f_E^\sharp\|_p$. This completes the proof. \square

Lemma 2.3 (van der Corput) *Let $I = (a, b)$ be a bounded interval on the real line and $k \geq 1$ an integer. Suppose $\phi \in C^k(I)$ is real-valued and satisfies one of the following conditions:*

- (i) $k = 1$, $|\phi'(t)| \geq 1$ for all $t \in I$ and ϕ' is monotone on I ;
- (ii) $k \geq 2$, $|\phi^{(k)}(t)| \geq 1$ for all $t \in I$.

Then there exists a constant C , depending only on k but not on I , such that

$$\left| \int_I e^{i\lambda\phi(t)} \varphi(t) dt \right| \leq C|\lambda|^{-1/k} \left(|\varphi(b)| + \int_I |\varphi'(t)| dt \right)$$

for $\lambda \in \mathbb{R}$ and $\varphi \in C^1[a, b]$.

For the proof of this lemma and related topics, one can see [29] and [4].

3 Certain Fractional Integrals

We shall see that L^p boundedness of T in (1.7) has some connections to certain fractional integrals of Hilbert type. In particular, when S is a monomial, the endpoint estimates will rely on properties of a simple class of fractional integral operators. In this section, we shall establish some mapping properties of these operators.

Theorem 3.1 *Let $W_{a,b}$ be the integral operator given by*

$$W_{a,b}f(x) = \int_{-\infty}^{\infty} ||x|_a^a - |y|_a^a|^{-\frac{1}{b}} f(y) dy$$

with $b \geq a > 1$, then $W_{a,b}$ is bounded from L^p to L^q for $\frac{1}{p} = \frac{1}{q} + \frac{b-a}{b}$ and $1 < p < \frac{b}{b-a}$.

Proof. Observe that the integral kernel is homogeneous of order $-a/b$. If $b = a > 1$, then we can use the Minkowski inequality to prove the statement by a change of variables. For $b \geq a$, we obtain

$$\|W_{a,b}f\|_q \leq \| |f|^{1-\theta} \|_{b/(b-a)} \|W_{a,a}(|f|^{b\theta/a})\|_{aq/b}^{a/b} \leq C \|f\|_{b(1-\theta)/(b-a)}^{1-\theta} \|f\|_{q\theta}^{\theta},$$

with $b/a < q < \infty$. If we set $p = b(1-\theta)/(b-a) = q\theta$, i.e. $\theta = p/q$, the desired inequality follows. \square

Now we apply above theorem to establish (L^p, L^q) estimates for oscillatory integral operators with polynomial phases.

Theorem 3.2 *Suppose that S is a real-valued homogeneous polynomial given by (1.2) and k_{\min} as in (1.4). If $k_{\min} \leq n/2$, then T defined by (1.7) extends as a bounded operator from $L^{2(n-k_{\min})/(2n-3k_{\min})}$ to L^2 .*

Proof. If $n = 2$, T reduces to the Fourier transform and hence T is bounded on L^2 by Plancherel's theorem. Now assume $n > 2$. For any $r > 0$, an application of the van der Corput lemma yields

$$\begin{aligned} \int_{-r}^r |Tf(x)|^2 dx &= \int_{-r}^r \int_{\mathbb{R}} \int_{\mathbb{R}} \exp\left(ia_{k_{\min}}x^{n-k_{\min}}(y^{k_{\min}} - z^{k_{\min}}) + Q(x, y, z)\right) f(y)\overline{f(z)} dy dz dx \\ &\leq C \int_{\mathbb{R}} \int_{\mathbb{R}} \left| |y|^{k_{\min}} - |z|^{k_{\min}} \right|^{-1/(n-k_{\min})} |f(y)f(z)| dy dz \\ &\leq C \|f\|_{2(n-k_{\min})/(2n-3k_{\min})}^2, \end{aligned}$$

where the last inequality follows from the Hardy-Littlewood-Sobolev inequality if $k_{\min} = 1$ and from Theorem 3.1 if $k_{\min} > 1$. Letting $r \rightarrow \infty$, we conclude the proof. \square

Let S be a real-valued homogeneous polynomial in $\mathbb{R} \times \mathbb{R}$ as in (1.2) with degree $n \geq 3$. Then its partial derivative $S''_{xy}(x, y)$ can be written as

$$S''_{xy}(x, y) = cx^{\gamma}y^{\beta} \prod_{j=1}^m (y - \alpha_j x)^{m_j} \prod_{j=1}^s Q_j(x, y), \quad (3.12)$$

where $\alpha_j \neq 0$ are distinct real numbers and Q_j are positive definite quadratic forms. It is clear that $\gamma = n - k_{\max} - 1$ and $\beta = k_{\min} - 1$.

In the proof of our main results, the main step is to construct an analytic family of operators in a strip and then show these operators satisfying suitable estimates at the boundary of the strip considered. These operators are closely related to the Hessian S''_{xy} of S . As Theorem 1.3 shown, the damping factor $|S''_{xy}|$ with suitable power gives us sharp estimates on L^2 . However, it is not true generally for the endpoint estimate $H_E^1 - L^1$. We shall see that this difference depends on whether $\beta = 0$ or not in §5. Assume $\beta = 0$. Then the following theorem gives mapping properties of the integral operator with kernels $|S''_{x,y}|^z$ with $Re(z) = -1/(n-2)$ if either (i) $\gamma > 0$ and $m + s \geq 1$ or (ii) $\gamma = 0$ and $m + s \geq 2$.

Theorem 3.3 *Suppose that K is a measurable function on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying*

$$|K(x, y)| \leq c |x|^{-\theta_0 n} \prod_{k=1}^m |x - \alpha_k y|^{-\theta_k n} \quad (3.13)$$

and

$$|\nabla_y K(x, y)| \leq C |x|^{-\theta_0 n} \sum_{k=1}^m |x - \alpha_k y|^{-\theta_k n - 1} \prod_{j \neq k} |x - \alpha_j y|^{-\theta_j n}, \quad (3.14)$$

where $0 \leq \theta_0 < 1$, $0 < \theta_k < 1$ satisfy $\theta_0 + \sum_{k=1}^m \theta_k = 1$ and $\alpha_1, \dots, \alpha_m$ are distinct nonzero numbers. Let T_K be the integral operator given by

$$T_K f(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy. \quad (3.15)$$

Then T_K has a bounded extension from L^p to itself for $1 < p < \theta_0^{-1}$ and is also bounded from H^1 to L^1 .

Proof. The assumptions imply that either $m \geq 1$ in the case $\theta_0 > 0$ or $m \geq 2$ when $\theta_0 = 0$. It suffices to prove the theorem in the case $0 < \theta_0 < 1$ since the treatment of other cases is similar. To prove the L^p boundedness of T_K for $1 < p < \theta_0^{-1}$, we observe that $|K(x, y)| \leq \prod_{l=1}^n K_l(x_l, y_l)$, where $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ and $K_l(x_l, y_l) = |x_l|^{-\theta_0} \prod_{k=1}^m |x_l - \alpha_k y_l|^{-\theta_k}$. It follows that

$$|T_K f(x)| \leq \int_{\mathbb{R}^n} \prod_{l=1}^n K_l(x_l, y_l) |f(y)| dy. \quad (3.16)$$

By Minkowski's inequality, we can reduce higher dimension $n \geq 2$ to dimension one $n = 1$. For $n = 1$, by a change of variables, we obtain

$$\begin{aligned} \left(\int_{\mathbb{R}} |T_K f(x)|^p dx \right)^{1/p} &\leq \left\{ \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |x|^{-\theta_0} \prod_{k=1}^m |x - \alpha_k y|^{-\theta_k} |f(y)| dy \right)^p dx \right\}^{1/p} \\ &\leq \left(\int_{\mathbb{R}} |y|^{-1/p} \prod_{k=1}^m |1 - \alpha_k y|^{-\theta_k} dy \right) \|f\|_p. \end{aligned}$$

Note $0 < \theta_0 < 1$, $\theta_0 + \sum_{k=1}^m \theta_k = 1$ and $1 < p < \theta_0^{-1}$. It is easy to see that the above integral with respect to y is finite. Consequently, we have $\|T_K f(x)\|_p \leq C \|f\|_p$ for all $1 < p < \theta_0^{-1}$.

Now we turn to prove that T_K is bounded from H^1 to L^1 . Suppose a is any given L^∞ atom in H^1 . Then there exists a cube Q such that (i) $\text{supp}(a) \subset Q$; (ii) $\|a\|_\infty \leq |Q|^{-1}$ and (iii)

$\int a(x)dx = 0$. It suffices to show that there exists a constant C , independent of a , such that $\|T_K(a)\|_{L^1} \leq C$. Let d_Q be the diameter of Q and c_Q its center. By dilation, we assume $|\alpha_k| \leq 1$ for $1 \leq k \leq m$. We shall now divide the proof into two cases. The first case is $|c_Q| \leq 2d_Q$. Then $|y| \leq 3d_Q$ for $y \in Q$. By the L^p boundedness of T_K , we apply Hölder's inequality to obtain $\|T_K(a)\|_{L^1(|x| \leq 5d_Q)} \leq C|Q|^{1-1/p} \|T(a)\|_{L^p} \leq C$ for $1 < p < \theta^{-1}$. When $|x| > 5d_Q$ and $y \in Q$, by the size assumption on $\nabla_y K$ and $|\alpha_k| \leq 1$, we obtain

$$\begin{aligned} |K(x, y) - K(x, c_Q)| &\leq Cd_Q |x|^{-\theta_0 n} \sup_{z \in Q} \left(\sum_{k=1}^n |x - \alpha_k z|^{-\theta_k n-1} \prod_{j \neq k} |x - \alpha_j z|^{-\theta_j n} \right) \\ &\leq Cd_Q |x|^{-n-1}. \end{aligned}$$

Hence it follows that

$$\int_{|x| > 5d_Q} |K(x, y) - K(x, c_Q)| dx \leq Cd_Q \int_{|x| > 5d_Q} |x|^{-n-1} dx < \infty.$$

Thus we have

$$\begin{aligned} \|T_K a\|_{L^1(|x| > 5d_Q)} &= \int_{|x| > 5d_Q} \left| \int_Q (K(x, y) - K(x, c_Q)) a(y) dy \right| dx \\ &\leq \left(\sup_{y \in Q} \int_{|x| > 5d_Q} |K(x, y) - K(x, c_Q)| dx \right) \int_Q |a(y)| dy \\ &\leq C. \end{aligned}$$

The second case is $|c_Q| > 2d_Q$. It is easy to see that $|c_Q|/2 \leq |y| \leq 3|c_Q|/2$ for all $y \in Q$. This observation implies

$$\sup_{y \in Q} \int_{|x| \leq 5|c_Q|} |K(x, y)| dx \leq C < \infty$$

by the assumption $0 < |\alpha_k| \leq 1$. Thus the integral of $|T_K(a)|$ over the ball $|x| \leq 5|c_Q|$ is bounded by a constant C . For $|x| \geq 5|c_Q|$, $|K(x, y) - K(x, c_Q)|$ has the same upper bound $Cd_Q |x|^{-n-1}$ uniformly for $y \in Q$. Hence the integral $|T_K(a)|$ over $|x| \geq 5|c_Q|$ is also less than a bound independent of a .

Combing above estimates, we conclude the proof of the theorem. \square

4 Oscillatory Integral Operators On H_E^1

Let P be a real-valued polynomial on $\mathbb{R}^n \times \mathbb{R}^n$. Let T_P be an oscillatory integral operator as in (1.9) with the kernel K as in Theorem 3.3. In this section, our purpose is to establish the boundedness of T_P from H_E^1 to L^1 . This result will serve as an endpoint estimate for an analytic family of operators in the proof of Theorem 1.1. If P can be written as $P(x, y) = P_1(x) + P_2(y)$, it is easy to see that T_P is bounded from H_E^1 to L^1 . In fact, H_E^1 is isomorphic to H^1 by a multiplication of $\exp(-iP_2(y))$.

Theorem 4.1 *Suppose that P is a real-valued polynomial and that T_P is given by (1.9) with K as in Theorem 3.3. Then T_P is a bounded operator from L^p to itself for $1 < p < \theta_0^{-1}$, and T_P has a bounded extension from $H_E^1(\mathbb{R}^n; P)$ to $L^1(\mathbb{R}^n)$. Moreover, the operator norms of T_P depend only on the degree of P but not on its coefficients.*

We first introduce a useful lemma which was essentially proved by Ricci and Stein in [24]. Some related topics were systematically studied in [4].

Lemma 4.2 *Let T_λ be the oscillatory integral operator defined by*

$$T_\lambda f(x) = \int_{\mathbb{R}^n} e^{i\lambda P(x,y)} \phi(x,y) f(y) dy,$$

where P is a real-valued polynomial with degree $d \geq 2$, $\phi \in C_0^\infty$ and $\lambda \in \mathbb{R}$. If there exist multi-indices β_1 and β_2 satisfying

$$\left| \frac{\partial^d P}{\partial^{\beta_1} x \partial^{\beta_2} y} \right| \geq 1, \quad (4.17)$$

with $0 < |\beta_1| < d$ and $|\beta_1| + |\beta_2| = d$, then there exists some $\delta > 0$, depending only on the degree d of P , such that the following decay estimate holds,

$$\|T_\lambda f\|_2 \leq C |\lambda|^{-\delta} \|f\|_2. \quad (4.18)$$

Proof. For $1 < p < \theta_0^{-1}$, the L^p boundedness of T_P follows immediately from Theorem 3.3, since we can take absolute value in the integral and the resulting operator is bounded on $L^p(\mathbb{R}^n)$. Now we turn our attention to prove the boundedness of T_P from H_E^1 to L^1 by induction on the degree d of P . For $d = 0$, this statement is just Theorem 3.3. For $d = 1$, $P(\cdot, \cdot)$ is degenerate in the sense that it can be decomposed as the sum of two polynomials $P_1(x)$ and $P_2(y)$. The statement is also true.

Assume the theorem is true for all polynomials of degree l not greater than $d - 1$. We shall prove that it is also true for $l = d$. Write

$$P(x, y) = \sum_{|\beta_1| + |\beta_2| = d} c_{\beta_1, \beta_2} x^{\beta_1} y^{\beta_2} + Q(x, y) = P_d(x, y) + Q(x, y)$$

with $\deg(Q) \leq d - 1$. Assume that P_d is not degenerate without loss of generality. For the same reason, we may assume that the coefficients of the pure x^α and y^β terms with $|\alpha| = |\beta| = d$ are zero. Given any H_E^1 atom a associated to P , the aim now is to show that $\|T_P a\|_1 \leq C$ for some $C < \infty$ independent of a . Let Q be the cube associated with a , with the center c_Q and diameter d_Q . For convenience, we divide the proof into two cases.

Case I. $|c_Q| \leq 2d_Q$.

Set $N = 10 + \max_k |\alpha_k|$. Since T_P is bounded on L^p for $1 < p < \theta_0^{-1}$, we obtain

$$\|Ta\|_{L^1(|x| \leq 5Nd_Q)} \leq C |Q|^{1-1/p} \|Ta\|_{L^p} \leq C.$$

For $|x| > 5Nd_Q$, we may write

$$\begin{aligned} T_P(a)(x) &= \int_Q e^{iP(x,y)} K(x,y) a(y) dy \\ &= \int_Q e^{iP(x,y)} \left(K(x,y) - K(x, c_Q) \right) a(y) dy + K(x, c_Q) \int_Q e^{iP(x,y)} a(y) dy \\ &= I_1 + I_2. \end{aligned}$$

Recall we have proved that in Theorem 3.3

$$\sup_{y \in Q} \int_{|x| > 5Nd_Q} |K(x,y) - K(x, c_Q)| dx \leq C < \infty \quad (4.19)$$

which implies

$$\int_{|x|>5Nd_Q} |I_1| dx \leq C \int_Q |a(y)| dy \leq C < \infty.$$

Before applying Lemma 4.2 to estimate I_2 , we need an additional argument to show that the bounds are independent of the coefficients of P . Recall that the coefficients of the pure x^α and y^β terms are assumed to be zero for all multi-indices $|\alpha| = |\beta| = d$. Decompose P as

$$P(x, y) = \sum_{|\alpha|+|\beta|=d} c_{\alpha,\beta} (x - c_Q)^\alpha (y - c_Q)^\beta + R(x, y),$$

where the degree of R is less than d . Then it is easy to see $P(c_Q, y) = R(c_Q, y)$. By the induction hypothesis, $\|T_R(a)\|_1 \leq C$. For $|x| > 5Nd_Q$, we can also write $T_R(a)$ as

$$T_R(a)(x) = \int_Q e^{iR(x,y)} \left(K(x, y) - K(x, c_Q) \right) a(y) dy + K(x, c_Q) \int_Q e^{iR(x,y)} a(y) dy$$

By (4.19), it follows that

$$\int_{|x|>5Nd_Q} \left| K(x, c_Q) \int_Q e^{iR(x,y)} a(y) dy \right| dx \leq C, \quad (4.20)$$

with the bound C independent of the coefficients of R . For $|x| > 5Nd_Q$, then $|x - c_Q| \geq 4Nd_Q$. It is convenient to observe that (4.20) is still true with $|x| > 5Nd_Q$ replaced by $|x - c_Q| \geq 4Nd_Q$. For $t > 0$, write

$$\begin{aligned} & \int_{4Nd_Q \leq |x - c_Q| \leq t} \left| K(x, c_Q) \int_Q e^{iP(x,y)} a(y) dy \right| dx \\ &= \int_{4Nd_Q \leq |x - c_Q| \leq t} \left| K(x, c_Q) \int_Q \left(e^{iP(x,y)} - e^{iR(x,y)} \right) a(y) dy \right| dx \\ & \quad + \int_{4Nd_Q \leq |x - c_Q| \leq t} \left| K(x, c_Q) \int_Q e^{iR(x,y)} a(y) dy \right| dx \\ &= I_{2,1} + I_{2,2}. \end{aligned}$$

It is clear that $|I_{2,2}| \leq C$ since (4.20) is still true with $|x| > 5Nd_Q$ replaced by $|x - c_Q| \geq 4Nd_Q$. Note that $|x - \alpha_k c_Q| \approx |x - c_Q|$ for $|x - c_Q| > 4Nd_Q$, we have

$$\begin{aligned} |I_{2,1}| &\leq \sum_{\alpha,\beta} |c_{\alpha,\beta}| \|a\|_{L^1} \int_{C_1 d_Q \leq |x - c_Q| \leq t} |x - c_Q|^{|\alpha|} |y - c_Q|^{|\beta|} |K(x, c_Q)| dx \\ &\leq \sum_{\alpha,\beta} |c_{\alpha,\beta}| d_Q^{d-|\alpha|} \int_{C_1 d_Q \leq |x - c_Q| \leq t} |x - c_Q|^{|\alpha| - n} dx \\ &\leq \sum_{\alpha,\beta} |c_{\alpha,\beta}| t^{|\alpha|} d_Q^{d-|\alpha|}, \end{aligned}$$

where the summations are taken over all multi-indices α and β satisfying $|\alpha| + |\beta| = d$ with $0 < |\alpha| < d$. Take

$$t^{-1} = \left(\max_{\alpha,\beta} |c_{\alpha,\beta}| d_Q^{d-|\alpha|} \right)^{1/|\alpha|} > 0,$$

where the maximum is taken over all multi-indices appearing in the above summations. It follows that $|I_{2,1}| \leq C$ with C depending only on the degree d and the dimension n . It remains to show that

$$\int_{|x-c_Q| \geq \max\{t, 4Nd_Q\}} \left| K(x, c_Q) \int_Q e^{iP(x,y)} a(y) dy \right| dx \leq C. \quad (4.21)$$

We have pointed out that $|x - c_Q| \geq 4Nd_Q$ implies $|K(x, c_Q)| \approx |x - c_Q|^{-n}$. By this observation and Lemma 4.2, we can assume now that Q is the unit cube centered at the origin. Indeed, we have

$$\begin{aligned} & \int_{|x-c_Q| \geq \max\{t, 4Nd_Q\}} \left| K(x, c_Q) \int_Q e^{iP(x,y)} a(y) dy \right| dx \\ & \leq \sum_{j=0}^{\infty} \int_{2^{j-1}t \leq |x-c_Q| < 2^j t} |x - c_Q|^{-n} \left| \int_Q e^{iP(x,y)} a(y) dy \right| dx. \end{aligned}$$

Let Q_0 be the unit cube centered at the origin and l the side length of Q . By a change of variables, we obtain

$$\begin{aligned} & \int_{2^{j-1}t \leq |x| < 2^j t} |x|^{-n} \left| \int_{Q_0} e^{iP(x+c_Q, ly+c_Q)} l^n a(ly + c_Q) dy \right| dx \\ & \leq C \int_{|x| \leq 2} \left| \int_{Q_0} e^{iP(2^j tx + c_Q, ly+c_Q)} l^n a(ly + c_Q) dy \right| dx \\ & = C \|U_j(l^n a(l(\cdot) + c_Q))\|_{L^2(|x| \leq 2)} \end{aligned}$$

where U_j is the operator given by

$$U_j f(x) = \int_{Q_0} e^{iP(2^j tx + c_Q, ly+c_Q)} f(y) dy.$$

By the choice of t , there exist multi-indices α and β satisfying $0 < |\alpha|, |\beta| < d$ and $|\alpha| + |\beta| = d$ such that

$$\left| \frac{\partial^d}{\partial x^\alpha \partial y^\beta} P(2^j tx + c_Q, ly + c_Q) \right| \geq C 2^{j|\alpha|}$$

with the constant C depending only on n and α . By Lemma 4.2, the power decay property of $\|U_j\|_{L^2 \rightarrow L^2} \leq C 2^{-j\delta}$ for some $\delta > 0$ implies the desired estimate (4.21).

Case II. $|c_Q| > 2d_Q$.

By the size condition (3.13) imposed on the kernel K , it follows that

$$\sup_{y \in Q} \int_{|x| \leq 2N|c_Q|} |K(x, y)| dx \leq C < \infty$$

with the constant C independent of Q . Let $t^{-1} = \left(\max_{\alpha, \beta} |c_{\alpha, \beta}| d_Q^{d-|\alpha|} \right)^{1/|\alpha|} > 0$ as above. By a similar argument, it is true that

$$\int_{2N|c_Q| < |x| < t} \left| \int_Q e^{iP(x,y)} K(x, y) a(y) dy \right| dx \leq C < \infty.$$

At the same time, the integral of $|T_P(a)|$ over $|x| > 2N|c_Q|$ is not greater than

$$\begin{aligned} & \int_{|x| > 2N|c_Q|} \left| \int_Q e^{iP(x,y)} [K(x,y) - K(x,c_Q)] a(y) dy \right| dx \\ & + \int_{|x| > 2N|c_Q|} |K(x,c_Q)| \left| \int_Q e^{iP(x,y)} a(y) dy \right| dx \\ & = J_1 + J_2. \end{aligned}$$

To estimate J_1 , first observe that the assumption (3.14) of the kernel K implies

$$|K(x,y) - K(x,c_Q)| \leq Cd_Q |x|^{-n-1}$$

for all $|x| > 2N|c_Q|$. Hence we have $J_1 \leq C$. Now it remains to show that

$$\int_{|x| > \max\{t, 2N|c_Q|\}} |K(x,c_Q)| \left| \int_Q e^{iP(x,y)} a(y) dy \right| dx \leq C < \infty. \quad (4.22)$$

The proof is similar as above. Hence the theorem is completely proved. \square

The following theorem deals with a class of translation invariant operators.

Theorem 4.3 *Let*

$$Uf(x) = \int_{-\infty}^{\infty} e^{i(x-y)^n} (1 + |x-y|)^z f(y) dy, \quad (4.23)$$

where $n \geq 2$ is an integer and $\operatorname{Re}(z) = -1$. Then U is bounded on L^2 and maps H^1 boundedly into L^1 with the bounds less than a constant multiple of $(1 + |\operatorname{Im}(z)|)^2$.

As mentioned in the introduction, the theorem is essentially contained in [13] (at least for even n). The argument in [13] is also applicable here. We shall give a new proof of the L^2 boundedness at the end of §5. By the argument in the proof of above theorem, we can show that U is bounded from H^1 to L^1 .

5 Damped Oscillatory Integral Operators

In §4, we have considered a class of oscillatory integral operators with critical negative power and obtained the endpoint estimate from H_E^1 to L^1 . To put T defined by (1.7) into a family of analytic operators, we shall insert a damped factor into T . For this purpose, we shall study operators of the following form

$$Wf(x) = \int_{-\infty}^{\infty} e^{iS(x,y)} |D(x,y)|^z f(y) dy \quad (5.24)$$

with suitably chosen z and the damped factor D being determined by S''_{xy} .

In this section, our main result is Theorem 5.1. It serves as an endpoint estimate of the operator W . The other endpoint estimate for W has been obtained in §4 except some special cases. More precisely, when the Hessian is of form $S''_{xy} = cy^\beta(y - \alpha x)^{n-2-\beta}$, the treatment is different and we shall deal with this case separately. For related results about damped oscillatory integral operators, we refer the reader to [23] and [18]. A general class of weighted oscillatory integral operators had been studied in [18]. The region obtained in [18] is given by an infinite intersection whose boundary is obscure. The damped factor D and its critical damping exponent are explicitly given in this section.

Theorem 5.1 Assume the Hessian S''_{xy} as in (3.12). Let W be given by (5.24) with

$$D(x, y) = x^\gamma \prod_{j=1}^m (y - \alpha_j x)^{m_j} \prod_{j=1}^s Q_j(x, y) \quad (5.25)$$

and

$$\operatorname{Re}(z) = a_\beta = \frac{1}{2(\beta+1)} \frac{n-2(\beta+1)}{n-\beta-2}.$$

Then W extends as a bounded operator on $L^2(\mathbb{R})$ with the operator norm $\|W\|$ less than a constant multiple of $(1 + |\operatorname{Im}(z)|)^2$.

Remark. For $\beta = 0$, this result is the same as Theorem 1.3. When $\beta = (n-2)/2$ and $z = 0$, the statement is contained in Theorem 3.2 since $k_{\min} = n/2$ and $a_\beta = 0$. If $\beta = n-2$, a_β is not well defined. In this case, we put the damping factor $D \equiv 1$ and $|D|^z = 1$. By insertion of a smooth cut-off, we obtain

$$W_\lambda f(x) = \int_{\mathbb{R}} e^{i\lambda S(x,y)} |D(x,y)|^z \varphi(x,y) f(y) dy \quad (5.26)$$

for $\lambda \in \mathbb{R}$ and $\varphi \in C_0^\infty(\mathbb{R} \times \mathbb{R})$. By dilation, the L^2 boundedness is equivalent to the following decay estimate

$$\|W_\lambda f\|_{L^2} \leq C |\lambda|^{-\frac{1}{2(1+\beta)}} \|f\|_{L^2}. \quad (5.27)$$

Our proof uses the techniques introduced in [21], [22] and [23]. By a decomposition of the singular variety $\{S''_{xy}(x,y) = 0\}$, we can write $W_\lambda = \sum_{k,l \geq 0} W_{k,l}$. By the almost orthogonality principle, we shall balance size and oscillatory estimates to obtain the sharp L^2 decay rate. The oscillatory estimates rely on the following operator version of van der Corput lemma.

Lemma 5.2 ([22]) Assume T_λ and S as in the introduction. Suppose that $\phi(x)$ is monotone and continuous on the interval $[\alpha, \beta]$. For some $\delta > 0$, the cut-off function φ and the phase S satisfy

$$(i) \quad \operatorname{supp}(\varphi) \subset \{(x, y) : \phi(x) \leq y \leq \phi(x) + \delta, \alpha \leq x \leq \beta\};$$

$$(ii) \quad \left| \partial_y^k \varphi(x, y) \right| \leq C \delta^{-k} \text{ for } k = 0, 1, 2;$$

(iii) $\mu \leq |S''_{xy}| \leq C\mu$ on the curved box $\{(x, y) : \phi(x) \leq y \leq \phi(x) + \delta, \alpha \leq x \leq \beta\}$ for some $\mu > 0$. Then

$$\|T_\lambda f\|_{L^2} \leq C |\lambda \mu|^{-1/2} \|f\|_{L^2}$$

with the bound independent of f .

We also need a lemma which measures the orthogonality in the decomposition of the operator W_λ . Results of this type are of fundamental importance in the study of damped oscillatory integral operators; see [23]. The following lemma is a simplified version of those given in [23].

Let T_1 and T_2 be two oscillatory integral operators given by

$$T_j f(x) = \int_{-\infty}^{\infty} e^{i\lambda S(x,y)} \varphi_j(x,y) f(y) dy, \quad j = 1, 2 \quad (5.28)$$

where S is a real-valued homogeneous polynomial given as in (1.2) and $\varphi_j \in C_0^\infty$. Suppose φ_j is supported in a parallelogram which has two sides parallel to the y -axis with length δ_j for each j . Denote these two parallelograms by Ω_1 and Ω_2 . Then

$$\Omega_j = \{(x, y) : a_j \leq x \leq b_j, \quad l_j(x) \leq y \leq l_j(x) + \delta_j\}$$

where $y = l_j(x)$ is a line in the plane.

Lemma 5.3 *Let T_1 and T_2 be defined as above. Let Ω_1^* be the expanded parallelogram*

$$\Omega_1^*(c) = \{(x, y) : a_1 \leq x \leq b_1, \quad l_1(x) - c\delta_1 \leq y \leq l_1(x) + (1+c)\delta_1\}$$

with $c > 0$. If there exist positive numbers c_1 and c_2 with $c_1 < c_2$ such that $\Omega_2 \subset \Omega_1^(c_2)$, the Hessian S''_{xy} does not change sign in $\Omega_1^*(c_2)$ and*

$$\min_{\Omega_1^*(c_1)} |S''_{xy}| \geq \mu > 0 \quad \text{and} \quad \max_{\Omega_1^*(c_2)} |S''_{xy}| \leq C\mu,$$

then we have

$$\|T_1 T_2^*\|_{L^2 \rightarrow L^2} \leq C(|\lambda|\mu)^{-1} \prod_{j=1}^2 \left(\max_{\Omega_j} \sum_{k=0}^2 \delta_j^k |\partial_y^k \varphi_j(x, y)| \right).$$

Remark. If Ω_1 and Ω_2 are two rectangles with sides parallel to the axes, the above assumptions can be relaxed. The assumptions in the lemma can be replaced by the following conditions:

(i) Let $\Omega_1^*(c)$ be the expanded rectangle only in the x -dimension but with the y -dimension unchanged,

$$\Omega_1^*(c) = \{(x, y) : a_1 - c(b_1 - a_1) \leq x \leq b_1 + c(b_1 - a_1), \quad \alpha_1 \leq y \leq \beta_1\}$$

with $c > 0$. Then assume that

$$\min_{\Omega_1^*(c_1)} |S''_{xy}| \geq \mu > 0, \quad \Omega_2 \subset \Omega_1^*(c_2)$$

for some $c_1 > 0$ and $c_2 > 0$;

(ii) Let \mathcal{R} be the rectangle consisting of all segments joining two points $(x_1, y) \in \Omega_1$ and $(x_2, y) \in \Omega_2$. The Hessian S''_{xy} does not change sign on \mathcal{R} and $\max_{\mathcal{R}} |S''_{xy}| \leq C\mu$.

The proof of above two lemmas relies on a basic property of polynomials. More precisely, for an arbitrary given polynomial P in \mathbb{R} , we have

$$\sup_{x \in I^*} |P^{(k)}(x)| \leq C|I|^{-k} \sup_{x \in I} |P(x)|$$

where the bound C depends only on the degree of P but not on the choice of the interval I and I^* is the interval concentric with I but dilated by the factor 2. A general concept of polynomial-like functions was introduced in [22].

We also use Schur's lemma frequently. For convenience, we state it as follows.

Lemma 5.4 *Suppose that $K(\cdot, \cdot)$ is measurable in $\mathbb{R} \times \mathbb{R}$ satisfying*

$$\sup_x \int |K(x, y)| dy \leq A \quad \text{and} \quad \sup_y \int |K(x, y)| dx \leq B$$

for some $A, B > 0$. Then the integral operator with kernel K is bounded on L^2 with bound not greater than \sqrt{AB} .

With above preliminaries, we turn to the proof of Theorem 5.1. The symbol $\|V\|$ means the norm of the operator V on L^2 . We also use $A \approx B$ to mean that $C_1 A \leq B \leq C_2 A$ for some constants $C_1, C_2 > 0$.

Proof. Choose a smooth function Φ satisfying $\text{supp } \Phi \subset [1/2, 2]$ and $\sum_{j \in \mathbb{Z}} \Phi(x/2^j) = 1$ for all $x > 0$. If the support of φ is sufficiently small, then we can divide W_λ in (5.26) into four parts of the form

$$\int_{\mathbb{R}} e^{i\lambda S(x,y)} |D(x,y)|^z \chi_{\{\pm x > 0\}}(x) \chi_{\{\pm y > 0\}}(y) \varphi(x,y) f(y) dy,$$

where χ_A is the characteristic function of the set A .

We shall prove that each of above four operators satisfies the desired estimate (5.27). Since the argument is similar, it suffices to show that the desired estimate holds for one of these operators. For the estimate of W_λ in the first quadrant, i.e., $x > 0, y > 0$, decompose W_λ as

$$W_\lambda f(x) = \sum_{k,l \geq 0} W_{k,l} f(x),$$

where

$$W_{k,l} f(x) = \int_{\mathbb{R}} e^{i\lambda S(x,y)} |D(x,y)|^z \Phi_k(x) \Phi_l(y) \varphi(x,y) f(y) dy \quad (5.29)$$

with $\text{Re}(z) = a_\beta$ and $\Phi_k(x) = \Phi(2^k x)$. We shall use $W_{k,l}^{++}$ to denote the decomposition in the first quadrant. Likewise, $W_{k,l}^{\sigma_1, \sigma_2}$ is defined similarly with $\sigma_1, \sigma_2 = \pm$.

Range $k \geq l + N_0$.

Assume $N_0 > 0$ is a large number such that $x \approx 2^{-k}, y \approx 2^{-l}$ imply $|y - \alpha_i x| \approx 2^{-l}$ if $k \geq l + N_0$. In this case, the proof is somewhat different depending on whether γ equals 0 or not. We begin with the simpler case $\gamma = 0$. It is convenient to estimate the summation of operators $W_{k,l}^{++} + W_{k,l}^{-+}$ in the range $k \geq l + N_0$. For each $l \geq 0$, let $U_l = \sum_{k \geq l + N_0} (W_{k,l}^{++} + W_{k,l}^{-+})$. By the support of Φ , we see that $U_l U_{l'}^* = 0$ unless $|l - l'| \leq 1$. For $U_l^* U_{l'}$, we claim that $\|U_l^* U_{l'}\|$ is bounded by $C_z |\lambda|^{-\frac{1}{\beta+1}} 2^{-|l-l'|\delta}$ with some $\delta > 0$. Observe that U_l is supported in the rectangle $|x| \leq 2^{-N_0+1} 2^{-l}$ and $y \approx 2^{-l}$. Without loss of generality, we may assume $l \leq l'$. It is easy to verify that the assumptions in the remark after Lemma 5.3 are satisfied with the roles x and y reversed. Hence we have

$$\|U_l^* U_{l'}\| \leq A = C(1 + |\text{Im}(z)|)^4 (|\lambda| 2^{-l(n-2)})^{-1} 2^{-l'(n-2-\beta)a_\beta} 2^{-l(n-2-\beta)a_\beta}.$$

By Schur's lemma, it follows that

$$\|U_l^* U_{l'}\| \leq B = C 2^{-l} 2^{-l'} 2^{-l(n-2-\beta)a_\beta} 2^{-l'(n-2-\beta)a_\beta}.$$

Combing above two estimates, we see that $\|U_l^* U_{l'}\|$ is not greater than $A^\theta B^{1-\theta}$ for all $0 \leq \theta \leq 1$. Taking $\theta = 1/(\beta+1)$, we obtain

$$\|U_l^* U_{l'}\| \leq C(1 + |\text{Im}(z)|)^{4\theta} |\lambda|^{-\frac{1}{\beta+1}} 2^{-|l-l'|(n-2)/[2(\beta+1)]}.$$

By the almost orthogonality lemma, we see that $\|\sum_l U_l\|$ is not greater than a constant multiple of $(1 + |\text{Im}(z)|)^2 |\lambda|^{-\frac{1}{2(\beta+1)}}$.

Now assume $\gamma > 0$. Denote by $k \wedge k'$ the minimum of k and k' . Then $W_{k,l}W_{k',l'}^*$ with $|l - l'| \leq 1$ satisfies the following estimates,

$$\begin{aligned}\|W_{k,l}W_{k',l'}^*\| &\leq C(1 + |\operatorname{Im}(z)|)^4 \left(|\lambda| 2^{-(k \wedge k')\gamma} 2^{-l(n-2-\gamma)} \right)^{-1} 2^{-(k+k')\gamma a_\beta} 2^{-(l+l')(n-2-\beta-\gamma)a_\beta} \\ \|W_{k,l}W_{k',l'}^*\| &\leq C 2^{-(k+k')/2} 2^{-(l+l')/2} 2^{-(k+k')\gamma a_\beta} 2^{-(l+l')(n-2-\beta-\gamma)a_\beta}.\end{aligned}$$

A convex combination of above estimates gives

$$\|W_{k,l}W_{k',l'}^*\| \leq C(1 + |\operatorname{Im}(z)|)^{4/(\beta+1)} |\lambda|^{-\frac{1}{\beta+1}} 2^{-|k-k'|\delta}$$

with $\delta = \gamma a_\beta + \beta/[2(\beta+1)] > 0$. Indeed, it is easy to verify that

$$\delta = \frac{1}{2(\beta+1)} \frac{(n-2)(\beta+\gamma) - (\beta+\gamma)^2 + \gamma^2}{n-2-\beta} > 0.$$

Similarly, by reversing x and y in Lemma 5.3, it is also true that for $|k - k'| \leq 1$

$$\|W_{k,l}^*W_{k',l'}\| \leq C(1 + |\operatorname{Im}(z)|)^{4/(\beta+1)} |\lambda|^{-\frac{1}{\beta+1}} 2^{-|l-l'|\delta}$$

with $\delta = (n + \beta - 2\gamma - 2)/[2(\beta+1)] + \gamma a_\beta$. We see $\delta > 0$ unless $\gamma = n - 2$. In the case $\gamma = n - 2$, $\delta = 0$ and we shall treat it separately. We shall rather estimate $W_k = \sum_{l=-\infty}^{\infty} (W_{k,l}^{++} + W_{k,l}^{+-})$. Also $W_k^*W_{k'} = 0$ if $|k - k'| \geq 2$. By the oscillatory estimate, we obtain

$$\|W_kW_{k'}^*\| \leq C(1 + |\operatorname{Im}(z)|)^4 |\lambda|^{-1} 2^{-|k-k'|(n-2)/2}.$$

Range $l \geq k + N_0$.

Now we turn to the estimate of $\|\sum W_{k,l}\|$ with summation being taken over $k \leq l - N_0$ for a sufficiently large $N_0 > 0$. For arbitrary two pairs (k, l) and (k', l') in the range, we also have, $|l - l'| \leq 1$,

$$\begin{aligned}\|W_{k,l}W_{k',l'}^*\| &\leq C 2^{-(k+k')/2} 2^{-(l+l')/2} 2^{-(k+k')(n-2-\beta)a_\beta} \\ \|W_{k,l}W_{k',l'}^*\| &\leq C(1 + |\operatorname{Im}(z)|)^4 \left(|\lambda| 2^{-l\beta} 2^{-(k \wedge k')(n-2-\beta)} \right)^{-1} 2^{-(k+k')(n-2-\beta)a_\beta}.\end{aligned}$$

By a convex combination of the above two inequalities, we obtain

$$\|W_{k,l}W_{k',l'}^*\| \leq C(1 + |\operatorname{Im}(z)|)^{4/(\beta+1)} |\lambda|^{-\frac{1}{\beta+1}} 2^{-|k-k'|\delta}$$

with $\delta = (n - \beta - 2)/[2(\beta+1)]$. By the same argument, we also have

$$\|W_{k,l}^*W_{k',l'}\| \leq C(1 + |\operatorname{Im}(z)|)^{4/(\beta+1)} |\lambda|^{-\frac{1}{\beta+1}} 2^{-|l-l'|\beta/[2(\beta+1)]}, \quad |k - k'| \leq 1.$$

For the special case $\beta = 0$, we rather estimate $V_k = \sum_l (W_{k,l}^{++} + W_{k,l}^{+-})$ over the range $l \geq k + N_0$. The assumptions in the remark of Lemma 5.3 are satisfied by V_k and $V_{k'}$ and then the resulting estimate is given by $\|V_kV_{k'}^*\| \leq C(1 + |\operatorname{Im}(z)|)^{4/(\beta+1)} |\lambda|^{-\frac{1}{\beta+1}} 2^{-|k-k'|(n-2)/2}$. Observe that $V_k^*V_{k'} = 0$ for $|k - k'| \geq 2$. Hence $\|\sum V_k\| \leq C(1 + |\operatorname{Im}(z)|)^2 |\lambda|^{-\frac{1}{2(\beta+1)}}$ by the almost orthogonality principle.

Range $|k - l| \leq N_0$.

For $|k - l| \leq N_0$, we shall further decompose $W_{k,l}$ as

$$W_{k,l} = \sum_{d \geq 0} W_{k,l,d} = \sum_d \int_{\mathbb{R}} e^{i\lambda S(x,y)} |D(x,y)|^z \Phi_k(x) \Phi_l(y) \Phi_d(y - \alpha_1 x) \varphi(x,y) f(y) dy$$

with $\operatorname{Re}(z) = a_\beta$. Observe that $d \geq k - C$ for some constant $C > 0$. It is convenient to divide the summation $\sum W_{k,l,d}$ over $d \geq k - C$ into two parts $|k - d| \leq N_1$ and $d > k + N_1$ with N_1 sufficiently large. More precisely, we may assume N_1 is so large that $|y - \alpha_1 x| \approx 2^{-k}$ in the support of $W_{k,l,d}$ for $2 \leq i \leq m$. It is also clear that $W_{k,l,d} W_{k',l',d'}^* = 0$ for $|l - l'| \geq 2$. Assume $|l - l'| \leq 1$. Then

$$\|W_{k,l,d} W_{k',l',d'}^*\| \leq C 2^{-(d+d')} \left(2^{-dm_1} 2^{-k(n-2-m_1-\beta)}\right)^{a_\beta} \left(2^{-d'm_1} 2^{-k(n-2-m_1-\beta)}\right)^{a_\beta}$$

and $\|W_{k,l,d} W_{k',l',d'}^*\|$ is also less than or equal to

$$C_z \left(|\lambda| 2^{-(d \wedge d')m_1} 2^{-k(n-2-m_1)}\right)^{-1} \left(2^{-dm_1} 2^{-k(n-2-m_1-\beta)}\right)^{a_\beta} \left(2^{-d'm_1} 2^{-k(n-2-m_1-\beta)}\right)^{a_\beta}.$$

with $C_z = C(1 + |\operatorname{Im}(z)|)^4$. Taking a convex combination, we obtain

$$\begin{aligned} \|W_{k,l,d} W_{k',l',d'}^*\| &\leq C_z |\lambda|^{-\frac{1}{\beta+1}} 2^{-|d-d'|\delta} \\ &\leq C(1 + |\operatorname{Im}(z)|)^4 |\lambda|^{-\frac{1}{\beta+1}} 2^{-|d-d'|\delta} \end{aligned}$$

with $\delta = \beta/(\beta + 1) + m_1 a_\beta > 0$. A similar argument gives us that $\|W_{k,l,d}^* W_{k',l',d'}\|$ is also bounded by a constant multiple of $|\lambda|^{-\frac{1}{\beta+1}} 2^{-|d-d'|\delta}$ with $\delta = \beta/(\beta + 1) + m_1 a_\beta > 0$. By the almost orthogonality principle, it follows that

$$\left\| \sum W_{k,l,d} \right\| \leq C(1 + |\operatorname{Im}(z)|)^2 |\lambda|^{-\frac{1}{2(\beta+1)}}$$

where the summation is taken over all $k, l, d \geq 0$ satisfying $|k - l| \leq N_0$ and $d \geq k + N_1$.

For $1 \leq \omega \leq m$, we use $W_{k,l,d_1,\dots,d_\omega}$ to denote the operator obtained from $W_{k,l,d_1,\dots,d_{\omega-1}}$ by multiplying a factor $\Phi_{d_\omega}(y - \alpha_\omega x)$ in the cut-off of $W_{k,l,d_1,\dots,d_{\omega-1}}$. Generally, we may apply the same argument as above to obtain

$$\left\| \sum W_{k,l,d_1,\dots,d_\omega} \right\| \leq C(1 + |\operatorname{Im}(z)|)^2 |\lambda|^{-\frac{1}{2(\beta+1)}}, \quad 1 \leq \omega \leq m$$

where the summation is taken over all nonnegative integers $k, l, d_1, \dots, d_\omega$ satisfying $|k - l| \leq N_0$, $|d_i - k| \leq N_i$ with $1 \leq i \leq \omega - 1$ and $d_\omega \geq k + N_\omega$, and N_1, \dots, N_ω are sufficiently large integers depending on the parameters α_i appearing in the factorization of S''_{xy} . Thus it remains to show that

$$\left\| \sum W_{k,l,d_1,\dots,d_m} \right\| \leq C(1 + |\operatorname{Im}(z)|)^2 |\lambda|^{-\frac{1}{2(\beta+1)}}$$

for $|k - l| \leq N_0$ and $|k - d_i| \leq N_i$ with $1 \leq i \leq m$. Observe that $W_{k,l,d_1,\dots,d_m} W_{k',l',d'_1,\dots,d'_m}^* = 0$ if $|l - l'| \geq 2$ and $W_{k,l,d_1,\dots,d_m}^* W_{k',l',d'_1,\dots,d'_m} = 0$ if $|k - k'| \geq 2$. By the almost orthogonality lemma,

it is enough to show that $\|W_{k,l,d_1,\dots,d_m}\|$ is uniformly bounded above by $C_z|\lambda|^{-\frac{1}{2(\beta+1)}}$. Indeed, we have

$$\begin{aligned}\|W_{k,l,d_1,\dots,d_m}\| &\leq C_z \min \left\{ \left(|\lambda| 2^{-k(n-2)} \right)^{-1/2} 2^{-k(n-2-\beta)a_\beta}, 2^{-k} 2^{-k(n-2-\beta)a_\beta} \right\} \\ &\leq C(1 + \operatorname{Im} |z|)^2 |\lambda|^{-\frac{1}{2(\beta+1)}}.\end{aligned}$$

The proof of the theorem is complete. \square

As pointed out at the beginning of this section, Theorem 5.1 should be adapted in some cases for our purpose. Now we turn our attention to the special case when S''_{xy} is of the form $cy^\beta(y - \alpha x)^{n-2-\beta}$. In the next section, we will see that Theorem 5.1 is not suitable for interpolation. Thus the damping factor shall be replaced by another one.

Theorem 5.5 *Assume S is a real-valued homogeneous polynomial in two variables with degree $n \geq 3$. If the Hessian S''_{xy} equals $cy^\beta(y - \alpha x)^{n-2-\beta}$ with $0 < \beta < (n-2)$ and nonzero numbers c and α , then the conclusion of Theorem 5.1 is also valid with the damping factor in (5.25) replaced by*

$$D(x, y) = x(y - \alpha x)^{n-3-\beta}. \quad (5.30)$$

Proof. Our argument is the same as that of above theorem. Let W_λ be defined as in (5.26) with the new damping factor $D(x, y)$. We can decompose W_λ as $\sum_{k,l \geq 0} W_{k,l}$ with $W_{k,l}$ given by (5.29). By dilation, we may assume $\alpha = 1$.

Estimates in the range $k \geq l + 10$.

Assume $|l - l'| \leq 1$. The size estimates of $W_{k,l} W_{k',l'}^*$ are given by

$$\|W_{k,l} W_{k',l'}^*\| \leq C 2^{-(k+l)/2} 2^{-(k'+l')/2} \left(2^{-k} 2^{-l(n-3-\beta)} \right)^{a_\beta} \left(2^{-k'} 2^{-l'(n-3-\beta)} \right)^{a_\beta}$$

and the oscillatory estimates assert that $\|W_{k,l} W_{k',l'}^*\|$ is bounded by

$$C(1 + |\operatorname{Im}(z)|)^4 \left(|\lambda| 2^{-l(n-2)} \right)^{-1} \left(2^{-k} 2^{-l(n-3-\beta)} \right)^{a_\beta} \left(2^{-k'} 2^{-l'(n-3-\beta)} \right)^{a_\beta}.$$

By a convex combination of above estimates, we obtain

$$\|W_{k,l} W_{k',l'}^*\| \leq C_z |\lambda|^{-\frac{1}{\beta+1}} 2^{-|k-k'|\delta}$$

with $\delta = \beta/[2(\beta+1)] + a_\beta > 0$. In fact, we have

$$\delta = \frac{1}{2(\beta+1)} \left(\beta + 1 - \frac{\beta}{n-2-\beta} \right) > 0.$$

Similarly, it is also true that

$$\|W_{k,l}^* W_{k',l'}\| \leq C_z |\lambda|^{-\frac{1}{\beta+1}} 2^{-|l-l'|\delta}$$

with $|k - k'| \leq 1$ and $\delta = a_\beta + (n + \beta - 2)/[2(\beta + 1)] > 0$.

Estimates in the range $l \geq k + 10$.

For $l \geq k + 10$ and $l' \geq k' + 10$ with $|l - l'| \leq 1$, we also have

$$\begin{aligned}\|W_{k,l}W_{k',l'}^*\| &\leq C2^{-(k+l)/2}2^{-(k'+l')/2}2^{-(k+k')(n-2-\beta)a_\beta} \\ \|W_{k,l}W_{k',l'}^*\| &\leq C_z \left(|\lambda|2^{-l\beta}2^{-(k \wedge k')(n-2-\beta)} \right)^{-1} 2^{-(k+k')(n-2-\beta)a_\beta}.\end{aligned}$$

Then it follows that $\|W_{k,l}W_{k',l'}^*\|$ is not greater than a constant multiple of $|\lambda|^{-\frac{1}{\beta+1}}2^{-|k-k'|\delta}$ with $\delta = (n - \beta - 2)/[2(\beta + 1)]$. A similar argument shows that $\|W_{k,l}^*W_{k',l'}\|$ is bounded by $C_z|\lambda|^{-\frac{1}{\beta+1}}2^{-|l-l'|\delta}$ with $\delta = \beta/[2(\beta + 1)] > 0$ for $|k - k'| \leq 1$.

Estimates in the range $|k - l| \leq 10$.

A further decomposition of $T_{k,l}$ is necessary to separate (x, y) from the line $x = y$ on which S''_{xy} vanishes. Let $T_{k,l,m}$ be defined as $T_{k,l}$ with the cut-off of $T_{k,l}$ multiplied by the factor $\Phi_m(x - y)$. It is clear that m is not less than $k - 12$. The treatment in this case is more direct and does not need the almost orthogonality principle. Actually, we shall see that the summation $\sum W_{k,l,m}$ is absolute convergent. The size of the support implies

$$\|W_{k,l,m}\| \leq C2^{-m} \left(2^{-k}2^{-m(n-\beta-3)} \right)^{a_\beta}.$$

The oscillatory estimate gives

$$\|W_{k,l,m}\| \leq C(1 + |\operatorname{Im}(z)|)^2 \left(|\lambda|2^{-k\beta}2^{-m(n-\beta-2)} \right)^{-1/2} \left(2^{-k}2^{-m(n-\beta-3)} \right)^{a_\beta}.$$

When $|\lambda| \geq 2^{k\beta}2^{m(n-\beta)}$, we take the oscillatory estimate as the bound of $\|W_{k,l,m}\|$. Observe that $m \geq k - 12$ in the present situation. For those k satisfying $2^{k\beta}2^{(k-12)(n-\beta)} \leq |\lambda|$, we use $m_{k,\lambda}$ to denote the largest integer m satisfying $2^{k\beta}2^{m(n-\beta)} \leq |\lambda|$. By this definition, it is easy to see that $|\lambda| \approx 2^{k\beta}2^{m_{k,\lambda}(n-\beta)}$. Now we observe that

$$\begin{aligned}\sum_{m \leq m_{k,\lambda}} \|W_{k,l,m}\| &\leq C_z \sum_{m \leq m_{k,\lambda}} \left(|\lambda|2^{-k\beta}2^{-m(n-\beta-2)} \right)^{-1/2} \left(2^{-k}2^{-m(n-\beta-3)} \right)^{a_\beta} \\ &\leq C_z \left(|\lambda|2^{-k\beta}2^{-m_{k,\lambda}(n-\beta-2)} \right)^{-1/2} \left(2^{-k}2^{-m_{k,\lambda}(n-\beta-3)} \right)^{a_\beta} \\ &\leq C_z |\lambda|^{-1/2} 2^{k(\beta/2-a_\beta)} \left(|\lambda|2^{-k\beta} \right)^{\delta_\beta}\end{aligned}$$

with $\delta_\beta = \frac{1}{n-\beta} \left(\frac{n-2-\beta}{2} - (n - \beta - 3)a_\beta \right)$. Recall that $2^{k\beta}2^{(k-12)(n-\beta)} \leq |\lambda|$. Hence

$$\begin{aligned}\sum_{2^{kn} \leq C|\lambda|} \sum_{m=k-13}^{m_{k,\lambda}} \|W_{k,l,m}\| &\leq C_z \sum_{2^{kn} \leq C|\lambda|} |\lambda|^{-1/2} 2^{k(\beta/2-a_\beta)} \left(|\lambda|2^{-k\beta} \right)^{\delta_\beta} \\ &\leq C(1 + |\operatorname{Im}(z)|)^2 |\lambda|^{-\frac{1}{2(\beta+1)}},\end{aligned}$$

where the exponent of 2^k equals

$$\frac{n}{n-\beta} \frac{\beta}{2(\beta+1)} - \frac{n}{n-\beta} \frac{n-2(\beta+1)}{2(\beta+1)} \frac{1}{n-\beta-2} > 0 \quad (5.31)$$

in the summation over $2^{kn} \leq C|\lambda|$. We must stress that the assumption $\beta > 0$ is crucial in the inequality (5.31). Similarly, we also have

$$\sum_{2^{kn} \leq C|\lambda|} \sum_{m=m_{k,\lambda}}^{\infty} \|W_{k,l,m}\| \leq C|\lambda|^{-\frac{1}{2(\beta+1)}}.$$

For those large k satisfying $2^{k\beta}2^{(k-12)(n-\beta)} > |\lambda|$, we use the size estimate as the upper bound of $\|W_{k,l,m}\|$. Then the desired estimate follows directly. Actually, the size estimate gives

$$\begin{aligned} \sum_{2^{kn} \geq C|\lambda|} \sum_{m \geq k-12} \|W_{k,l,m}\| &\leq C \sum_{2^{kn} \geq C|\lambda|} \sum_{m \geq k-13} 2^{-m} \left(2^{-k}2^{-m(n-\beta-3)}\right)^{a_\beta} \\ &\leq C \sum_{2^{kn} \geq C|\lambda|} 2^{-kn/[2(\beta+1)]} \\ &\leq C|\lambda|^{-\frac{1}{2(\beta+1)}}. \end{aligned}$$

The proof is therefore complete. \square

Remark. When $\beta = 0$, the inequality (5.31) is not true and its left side equals $-\frac{1}{2}$. Hence the argument above fails. We may ask whether Theorem 5.5 is still true in the case $\beta = 0$. The answer is negative. Actually, we would obtain that the following operator

$$Wf(x) = \int_{-\infty}^{\infty} e^{i(x-y)^n} |x(x-y)^{n-3}|^{1/2} f(y) dy$$

is bounded from L^2 to itself with $n \geq 3$ if Theorem 5.5 were still true when $\beta = 0$. Let f be the characteristic function of the interval $(N, N + \sqrt[n]{\pi/16})$ for $N \geq 1$. Then we see that $|Wf(x)| \geq CN^{1/2}$ for $N + \sqrt[n]{\pi/8} \leq x \leq N + \sqrt[n]{\pi/4}$. This observation implies that W is unbounded on L^2 . The following theorem is a substitute of the above theorem.

Theorem 5.6 *Let U be the operator as in Theorem 4.3 with $\operatorname{Re}(z) = (n-2)/2$. Then U extends as a bounded operator on L^2 with the operator norm not greater than a constant multiple of $(1 + |\operatorname{Im}(z)|)^2$.*

Remark. There is an equivalent formulation of the theorem. Assume $S(x, y) = c(x - \alpha y)^n$ for nonzero real numbers c and α . Let W_λ be defined as in (5.26) with $\operatorname{Re}(z) = (n-2)/2$ and the damping factor

$$D(x, y) = (|\lambda|^{-1/n} + |x - \alpha y|). \quad (5.32)$$

Then the estimate for W_λ in (5.27) is still true.

Proof. Assume $c = \alpha = 1$ without loss of generality. By the remark, it suffices to show that W_λ satisfies the estimate (5.27). Let $W_{k,l}^{\sigma_1, \sigma_2}$ be defined as in (5.29). In the case $k \geq l + 10$, let $U_l = \sum (W_{k,l}^{+,+} + W_{k,l}^{-,+})$ with the summation being taken over $k \geq l + 10$. It is easy to see that U_l is supported in the rectangle $|x| \leq C2^{-l}$ and $y \approx 2^{-l}$. By the Schur lemma, we obtain

$$\sum \|U_l\| \leq C \sum 2^{-l} |\lambda|^{-(n-2)/(2n)} \leq C|\lambda|^{-1/2}$$

where the summation is taken over $2^{ln} \geq |\lambda|$. For those l satisfying $2^{ln} < |\lambda|^{1/n}$, $U_l U_{l'}^* = 0$ for $|l - l'| \leq 1$ and it follows from Lemma 5.3 that $\|U_l^* U_{l'}\|$ is bounded by a constant multiple of

$(1 + |\operatorname{Im}(z)|)^4 |\lambda|^{-1} 2^{-|l-l'|(n-2)/2}$. The estimate in the case $l \geq k + 10$ can be treated similarly. Actually, set $V_k = \sum_{l \geq k+10} (W_{k,l}^{++} + W_{k,l}^{+-})$ for each k . Then we can divide the summation $\sum V_k$ into two parts, i.e. $2^{kn} \geq |\lambda|$ and $2^{kn} \leq |\lambda|$. The desired estimate follows by using the Schur lemma to the first part and the almost orthogonality to the second one.

Now consider the case $|k - l| \leq 10$. Define $W_{k,l,m}$ as $W_{k,l}$ with the cut-off of $W_{k,l}$ multiplied by the factor $\Phi_m(x - y)$. Then $W_{k,l,m} = 0$ unless $m \geq k - 13$. For those m satisfying $2^{mn} > |\lambda|$, let $U_m = \sum_{|k-l| \leq 10} W_{k,l,m}$ for each m . Then by Schur's lemma we obtain $\sum \|U_m\| \leq C|\lambda|^{-1/2}$ with the summation taken over all m satisfying $2^{mn} > |\lambda|$. For m satisfying $2^{mn} \leq |\lambda|$, we shall use the almost orthogonality. Indeed, we have

$$\begin{aligned} \|W_{k,l,m} W_{k',l',m'}^*\| &\leq C(1 + |\operatorname{Im}(z)|)^4 |\lambda|^{-1} 2^{-|m-m'|(n-2)/2}, \quad |l - l'| \leq 1 \\ \|W_{k,l,m}^* W_{k',l',m'}\| &\leq C(1 + |\operatorname{Im}(z)|)^4 |\lambda|^{-1} 2^{-|m-m'|(n-2)/2}, \quad |k - k'| \leq 1. \end{aligned}$$

Thus we have almost orthogonality which implies the desired estimate. \square

The following theorem implies the L^2 boundedness of U in Theorem 4.3.

Theorem 5.7 *Assume $n \geq 2$ is an integer. Let W_λ be given by*

$$W_\lambda f(x) = \int_{-\infty}^{\infty} e^{i\lambda(x-y)^n} \left(|\lambda|^{-1/n} + |x - y| \right)^z \varphi(x, y) f(y) dy \quad (5.33)$$

with $\operatorname{Re}(z) = -1$ and $\varphi \in C_0^\infty$. Then the operator norm $\|W_\lambda\|$ on L^2 is bounded by a constant multiple of $(1 + |\operatorname{Im}(z)|)^2$ with the constant independent of λ .

Proof. Define $W_{k,l}^{\sigma_1, \sigma_2}$ as in the proof of Theorem 5.1. Then we shall divide the proof into three cases $k \geq l + 10$, $l \geq k + 10$ and $|k - l| \leq 10$. In the case $k \geq l + 10$, define $U_l = \sum (W_{k,l}^{++} + W_{k,l}^{-+})$ for each l . For those l satisfying $2^{ln} \geq |\lambda|$, we apply Schur's lemma to obtain

$$\left\| \sum U_l \right\| \leq C \sum_{2^{ln} \geq |\lambda|} 2^{-l} |\lambda|^{1/n} \leq C.$$

For l and l' satisfying $2^{ln} \leq |\lambda|$ and $2^{l'n} \leq |\lambda|$, we invoke Lemma 5.3 to obtain

$$\|U_l^* U_{l'}\| \leq C_z (|\lambda| 2^{-(l \wedge l')(n-2)})^{-1} 2^{l+l'} \leq C_z 2^{-|l-l'|(n-1)}$$

with C_z independent of λ . Observe $U_l U_{l'}^* = 0$ for $|l - l'| \geq 2$. We have the almost orthogonality and then obtain $\|\sum U_l\| \leq C$. The treatment of the case $l \geq k + 10$ is similar. For $|k - l| \leq 10$, we shall introduce $W_{k,l,m}$ as above. For those m satisfying $2^{mn} \geq |\lambda|$, the desired estimate follows by the Schur's lemma. While for $2^{mn} \leq |\lambda|$, it is true that

$$\|W_{k,l,m} W_{k',l',m'}^*\| \leq C(1 + |\operatorname{Im}(z)|)^4 2^{-|m-m'|(n-1)}, \quad |l - l'| \leq 1.$$

The same upper bound is also valid for $\|W_{k,l,m}^* W_{k',l',m'}\|$ for $|k - k'| \leq 1$. By the almost orthogonality principle, we obtain $\|\sum W_{k,l,m}\| \leq C$. The proof is complete. \square

6 Proof of Theorem 1.1

In this section, we shall apply previous results to prove Theorem 1.1. Let p_0 and p_1 be equal to $n/(n - k_{\min})$ and $n/(n - k_{\max})$, respectively. The argument will be divided into two cases: (i) $k_{\min} < n/2$ and (ii) $k_{\min} \geq n/2$. In both cases, there may occur $k_{\min} = k_{\max}$, i.e., S is a monomial modulo pure- x and pure- y terms. We shall see that the following argument in the case (ii) is also applicable to the case when S is a monomial.

Proof. By duality, it is enough to show that T is bounded on L^{p_0} . If this were done, we would obtain that its adjoint operator T^* is bounded from $L^{n/k_{\max}}$ to itself. Thus T has a bounded extension from L^{p_1} to itself.

We first treat the case (ii) $k_{\min} \geq n/2$. Since T^* has the oscillating kernel $e^{-iS(y,x)}$, we see that the assumption $k_{\min} \geq n/2$ imposed on T is equivalent to the condition $k_{\max} \leq n/2$ on T^* . Hence it suffices to prove that T is bounded from $L^{n/(n-k_{\max})}$ to itself under the assumption $k_{\max} \leq n/2$. For brevity, we only prove that T maps $L^{n/(n-k_{\max})}(0, \infty)$ to itself boundedly since other parts of integration can be treated similarly.

Consider

$$H(f)(x) = \int_0^\infty e^{iS(x^{k_{\max}/(n-k_{\max})}, y)} f(y) dy$$

and denote by H^* its adjoint operator. By the van der Corput lemma, we have

$$\begin{aligned} \int_0^\infty |H^*(f)(x)|^2 dx &\leq C |a_{n-k_{\max}}|^{-1/k_{\max}} \int_0^\infty \int_0^\infty |y_1^{k_{\max}} - y_2^{k_{\max}}|^{-\frac{1}{k_{\max}}} |f(y_1)f(y_2)| dy_1 dy_2 \\ &\leq C |a_{n-k_{\max}}|^{-1/k_{\max}} \|f\|_2^2, \end{aligned}$$

where we impose the assumption on $k_{\max} > 1$. Otherwise if $k_{\max} = 1$, then H reduce to the Fourier transform. Hence H is bounded on $L^2(0, \infty)$. By Lemma 2.1, we have

$$\int_0^\infty |H(f)(x)|^p x^{p-2} dx \leq C \|f\|_p^p$$

for $1 < p \leq 2$. Set $p = n/(n - k_{\max})$. By a change of variables, we see that T is bounded on $L^p(0, \infty)$. When $k_{\min} = k_{\max}$, the range of p described in Theorem 1.1 is just a single value $p = n/(n - k_{\min})$. If $k_{\min} \geq n/2$, then we have proved that T is bounded on L^p . If $k_{\min} < n/2$, we can apply a duality argument to obtain the desired statement.

Now we turn to the case (i) $1 \leq k_{\min} < \frac{n}{2}$. Without loss of generality, we assume also $k_{\min} < k_{\max}$. The argument is somewhat different depending on whether the Hessian S''_{xy} is of the form $cy^\beta(y - \alpha x)^{n-\beta-2}$ with nonzero real numbers c and α . We shall divide the argument into three cases. Recall that S''_{xy} can be written as

$$S''_{xy}(x, y) = Cx^\gamma y^\beta \prod_{j=1}^m (y - \alpha_j x)^{m_j} \prod_{j=1}^s Q_j(x, y).$$

Since $k_{\min} < n/2$, it is easy to see $0 \leq \beta < (n - 2)/2$.

Case I: S''_{xy} is not of the form $cy^\beta(y - \alpha x)^{n-\beta-2}$. Then let D be the damping factor given by

$$D(x, y) = x^\gamma \prod_{j=1}^m (y - \alpha_j x)^{m_j} \prod_{j=1}^s Q_j(x, y). \quad (6.34)$$

Consider the following analytic family of operators

$$T_z f(x) = \int_{-\infty}^{\infty} e^{iS(x,y)} |D(x,y)|^z f(y) dy \quad (6.35)$$

for z in the strip $\left\{x + iy : -\frac{1}{n-2-\beta} \leq x \leq a_\beta\right\}$. It is clear that $T_0 = T$. Let $K_z(x, y) = |D(x, y)|^z$. Choose $\alpha_j \neq 0$ for $m+1 \leq j \leq m+2s$ such that $\alpha_j \neq \alpha_k$ for $1 \leq j \neq k \leq m+2s$. For $\operatorname{Re}(z) = -1/(n-2-\beta)$, the kernels K_z fall under the scope of the class discussed in Theorem 3.3 and Theorem 4.1. Indeed, for $\operatorname{Re}(z) = -1/(n-2-\beta)$, a direct computation shows that

$$|K_z(x, y)| \leq C|x|^{-\theta_0} \prod_{1 \leq j \leq m+2s} |x - c_j y|^{-\theta_j},$$

$$|\partial_y K_z(x, y)| \leq C|x|^{-\theta_0} \sum_{1 \leq k \leq m+2s} |x - c_k y|^{-\theta_k-1} \prod_{j \neq k} |x - c_j y|^{-\theta_j},$$

where

$$\theta_0 = \frac{\gamma}{n-2-\beta}, \quad \theta_j = \frac{m_j}{n-2-\beta}, \quad 1 \leq j \leq m \quad \text{and} \quad \theta_j = \frac{1}{n-2-\beta}, \quad m+1 \leq j \leq m+2s$$

with $c_j = \alpha_j^{-1} \neq 0$ for $1 \leq j \leq m+2s$. On the one hand, it follows from Theorem 4.1 that T_z is bounded from H_E^1 to L^1 with the bound less than a constant multiple of $(1 + |\operatorname{Im}(z)|)^2$. On the other hand, it follows from Theorem 5.1 that T_z is bounded on L^2 when $\operatorname{Re}(z) = a_\beta$ with the bound not greater than $C(1 + |\operatorname{Im}(z)|)^2$. Combing these results, we obtain

$$\|T_z^* f\|_{BMO_E} \leq A_z \|f\|_{L^\infty}$$

for $\operatorname{Re}(z) = -1/(n-2-\beta)$, and

$$\|T_z^* f\|_{L^2} \leq A_z \|f\|_{L^2}$$

for $\operatorname{Re}(z) = a_\beta$, where BMO_E is given by Definition 1.2 with $P(x, y) = -S(y, x)$. It follows immediately that

$$\left\| (T_z^* f)_E^\sharp \right\|_{L^\infty} \leq A_z \|f\|_{L^\infty} \quad (6.36)$$

for $\operatorname{Re}(z) = -1/(n-2-\beta)$, and

$$\left\| (T_z^* f)_E^\sharp \right\|_{L^2} \leq A_z \|f\|_{L^2} \quad (6.37)$$

for $\operatorname{Re}(z) = a_\beta$.

To apply the complex interpolation, we need a linear operator to approximate $(T_z^* f)_E^\sharp$. Let $\rho(x, y)$ be a measurable function with $|\rho(x, y)| \leq 1$. We use $Q(x)$ to denote a measurable mapping from x to a cube $Q(x)$ containing x . Then we can define a family A_z of operators by

$$A_z f(x) = \frac{1}{|Q(x)|} \int_{Q(x)} \left(T_z^* f(y) - (T_z^* f)_{Q(x)}^E \right) \rho(x, y) dy$$

with $(T_z^* f)_{Q(x)}^E$ defined as in the introduction. Then $(T_z^* f)_E^\sharp$ can be obtained by taking the supremum $\sup |A_z f(x)|$ over all $\rho(x, y)$ and $Q(x)$ described above. By interpolation and Lemma 2.2, we get that

$$\|T_z^*(f)\|_{L^p} \leq \left\| (T_z^*)^\sharp(f) \right\|_{L^p} \leq C_z \|f\|_{L^p}, \quad (6.38)$$

where p and z satisfy

$$\operatorname{Re}(z) = -\frac{1}{n-2-\beta}(1-\theta) + \alpha_\beta\theta$$

and $1/p = \theta/2$ for $0 \leq \theta \leq 1$. If $\operatorname{Re}(z) = 0$, we obtain $\theta = 2(\beta+1)/n$ and $p = n/(\beta+1)$. By duality, T_0 is bounded on $n/(n-(\beta+1))$. Since $k_{\min} = \beta+1$, the desired result follows.

Case II: $S''_{xy} = cy^\beta(y-\alpha x)^{n-\beta-2}$ with $0 < \beta < (n-2)/2$. Define the analytic family T_z as in (6.35) with the damping factor in (6.34) replaced by $D(x, y) = x(y-\alpha x)^{n-\beta-3}$. Then the above argument is also applicable here. Actually, when the real part of z is equal to a_β , T_z is bounded from L^2 to itself with the norm bounded by a constant multiple of $(1 + |\operatorname{Im}(z)|)^2$. For those z with $\operatorname{Re}(z) = -1/(n-2-\beta)$, we may invoke Theorem 4.1 to obtain that T_z is bounded from H_E^1 to L^1 . The interpolation argument is the same as above. Thus we show that T is bounded on L^{p_0} .

Case III: $S''_{xy} = c(y-\alpha x)^{n-2}$. By dilation, we may assume $\alpha = 1$. The damping factor shall be replaced by $D(x, y) = (1 + |x-y|)^{n-2}$. By combining Theorem 4.3 and Theorem 5.6, the desired result follows by a similar argument.

The proof is therefore complete. \square

7 Applications

In this section, we shall give two applications of Theorem 1.1. We begin with the proof of Theorem 1.2.

Proof. By Theorem 1.1 and Lemma 2.1, we have

$$\int_0^\infty \left| \int_0^\infty \exp(iS(x, y)) f(y) dy \right|^p x^{(p-q_0)/(q_0-1)} dx \leq C \int_0^\infty |f(x)|^p dx$$

for $n/(n-k_{\min}) \leq q_0 \leq n/(n-k_{\max})$ and $1 < p \leq q_0$.

Set

$$q_0 = p_0 = \frac{n}{n-k_{\min}} \quad \text{and} \quad p = \tilde{p}_0 = \frac{(n-k_{\min})m_1 + k_{\min}}{(n-k_{\min})m_1}.$$

It follows that

$$\int_0^\infty \left| \int_0^\infty \exp(iS(x^{m_1}, y)) f(y) dy \right|^p dx \leq C \int_0^\infty |f(x)|^p dx \quad (7.39)$$

with $p = \tilde{p}_0$. The above inequality also holds if we set

$$q_0 = p_1 = \frac{n}{n-k_{\max}} \quad \text{and} \quad p = \tilde{p}_1 = \frac{(n-k_{\max})m_1 + k_{\max}}{(n-k_{\max})m_1}.$$

By interpolation, we obtain that the inequality (7.39) is still true for $\tilde{p}_0 \leq p \leq \tilde{p}_1$. By a duality argument and the interpolation technique as above, we conclude that

$$\int_0^\infty \left| \int_0^\infty \exp(iS(x^{m_1}, y^{m_2})) f(y) dy \right|^p dx \leq C \int_0^\infty |f(x)|^p dx$$

holds for p in the range

$$\frac{(n-k_{\min})m_1 + k_{\min}m_2}{(n-k_{\min})m_1} \leq p \leq \frac{(n-k_{\max})m_1 + k_{\max}m_2}{(n-k_{\max})m_1}.$$

The other part of $T_{m_1, m_2} f$ can be treated similarly. Hence the L^p boundedness of T_{m_1, m_2} has been established. By dilation as in [22](also [31]), we can show that this range is also sharp. The proof is therefore complete. \square

Pitt's inequality is a generalized Hausdorff-Young inequality with power weights; see Beckner [3]. We shall give a simple proof by using previous results. Denote by \widehat{f} the Fourier transform of f in the Schwartz class,

$$\widehat{f}(x) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot y} f(y) dy.$$

Then Pitt's inequality can be stated as follows. For $1 \leq p \leq \infty$, let p' be the conjugate exponent of p , i.e. $1/p + 1/p' = 1$.

Theorem 7.1 ([3]) *Let $1 < p \leq q < \infty$, $0 \leq \alpha < n/q$ and $0 \leq \beta < n/p'$. There exists a constant $C > 0$ such that*

$$\left(\int_{\mathbb{R}^n} |x|^{-\alpha} |\widehat{f}(x)|^q dx \right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} |x|^\beta |f(x)|^p dx \right)^{1/p} \quad (7.40)$$

for all Schwartz functions f , where p, q, α and β satisfy $n/p + n/q + \beta - \alpha = n$.

Proof. We first prove the theorem in one dimension. For $n = 1$, by dividing the integration into two parts, it is enough to show that

$$\left(\int_0^\infty \left| \int_0^\infty e^{-2\pi i x y} f(y) dy \right|^q x^{-q\alpha} dx \right)^{1/q} \leq C \left(\int_0^\infty |f(x)|^p x^{p\beta} dx \right)^{1/p}. \quad (7.41)$$

Consider

$$Tg(x) = \int_0^\infty \exp(-2\pi i x^a y^b) g(y) dy,$$

for $a, b \geq 1$. By the same argument as in §6 when S is a monomial, we can show that T is bounded from $L^{(a+b)/a}(0, \infty)$ to itself. By dilation and interpolation, it is easy to see that T has a bounded extension from L^p to L^q with $1 \leq p \leq (a+b)/a$ and $1/p = 1 - b/(aq)$. Set $a = 1/(1 - q\alpha)$ and $b = 1/(1 - p'\beta)$. The inequality (7.41) follows by a change of variables and letting $g(y) = y^{b-1} f(y^b)$.

For dimension $n \geq 2$, observe that $|x|^{-\alpha} \leq \prod_{k=1}^n |x_k|^{-\alpha/n}$ and $|x|^\beta \geq \prod_{k=1}^n |x_k|^{\beta/n}$ for $x \in \mathbb{R}^n$ and $\alpha, \beta \geq 0$. Now we define \mathcal{F}_k to be the Fourier transform relative to x_k with other variables fixed. It is clear that \mathcal{F} is just the composition of \mathcal{F}_k , i.e., $\mathcal{F}f(x) = \mathcal{F}_n \mathcal{F}_{n-1} \cdots \mathcal{F}_1 f(x)$. By n applications of the one dimensional inequality and Minkowski's inequality, we obtain Pitt's inequality in dimension n . Thus the proof is complete. \square

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